DOI-KOPPINEN MODULES FOR QUANTUM GROUPOIDS

TOMASZ BRZEZIŃSKI, STEFAAN CAENEPEEL, AND GIGEL MILITARU

To Max Kelly on the occasion of his 70th birthday.

Abstract. A definition of a Doi-Koppinen datum over a noncommutative algebra is proposed. The idea is to replace a bialgebra in a standard Doi-Koppinen datum with a bialgebroid. The corresponding category of Doi-Koppinen modules over a noncommutative algebra is introduced. A weak Doi-Koppinen datum and module of [1] are shown to be examples of a Doi-Koppinen datum and module over an algebra. A coring associated to a Doi-Koppinen datum over an algebra is constructed and various properties of induction and forgetful functors for Doi-Koppinen modules over an algebra are deduced from the properties of corresponding functors in the category of comodules of a coring.

1. Introduction

Doi-Koppinen modules introduced in [11] [13] as a generalisation of (co)modules or Hopf modules studied in Hopf algebra theory can be viewed as a representation of a triple comprising an algebra, a coalgebra and a bialgebra which satisfy certain compatibility conditions. Recently these have been generalised to the case in which a bialgebra is replaced by a weak Hopf algebra [1]. It is known [12] that weak Hopf algebras are an example of a generalisation of a bialgebra known as an R-bialgebroid [14] or $\times_R$-bialgebra [20] (and leading to the notion of a Hopf algebroid or a quantum groupoid), introduced in the context of Poisson geometry, algebraic topology and classification of algebras. It seems therefore natural to ask whether a definition of a Doi-Koppinen datum in which a bialgebra is replaced by an R-bialgebroid is possible. In this paper we propose such a definition and by this means introduce the notion of a Doi-Koppinen module over a noncommutative algebra R. We show that Doi-Koppinen modules for a weak Hopf algebra are a special case thus providing a new, more general point of view on weak Doi-Koppinen data and modules.

On the other hand it has been realised in [4] that a natural point of view on Doi-Koppinen data is provided by entwining structures introduced in [6]. The same point of view was adopted in [8], where weak entwining structures were introduced in order to describe Doi-Koppinen data for a weak Hopf algebra. Later on it has been shown in [5] that both entwined modules and weak entwined modules are simply comodules of certain corings. Thus various properties of entwined modules such as Frobenius and separability properties discussed first in the case of Doi-Koppinen modules in [9] and [10], can be derived from the properties of comodules over a coring. In the present paper we show that a Doi-Koppinen datum for an R-bialgebroid leads to a certain coring whose comodules are precisely the Doi-Koppinen modules over R.

2. Preliminaries

2.1. **Notation.** We use the following conventions. For an object V in a category, the identity morphism $V \to V$ is denoted by V. All rings in this paper have 1, a ring map

2000 *Mathematics Subject Classification.* 16W30.

is assumed to respect 1, and all modules over a ring are assumed to be unital. For a ring R, $\mathcal{M}_R$ (resp. ${}_R\mathcal{M}$, ${}_R\mathcal{M}_R$) denotes the category of right R-modules (resp. left R-modules, R-bimodules). The action of R is denoted by a dot between elements.

Throughout the paper k denotes a commutative ring with unit. We assume that all the algebras are over k and unital, and coalgebras are over k and counital. Unadorned tensor product is over k. For a k-coalgebra C we use Δ_C to denote the coproduct and ϵ_C to denote the counit (we skip subscripts if no confusion is possible). Notation for comodules is similar to that for modules but with subscripts replaced by superscripts, i.e. ${}^C\mathcal{M}$ is the category of left C-comodules etc. We use the Sweedler notation for coproducts and coactions, i.e. $\Delta(c) = c_{(1)} \otimes c_{(2)}$ for a coproduct, and $\rho(m) = m_{<-1>} \otimes m_{<0>}$ for a left coaction (summation understood).

Let R be a k-algebra. Recall from [17] that an *R-coring* is a coalgebra in the monoidal category of R-bimodules $({}_R\mathcal{M}_R, \otimes_R, R)$, i.e., it is an (R,R)-bimodule $\mathcal{C}$ together with (R,R)-bimodule maps $\Delta_\mathcal{C} : \mathcal{C} \to \mathcal{C} \otimes_R \mathcal{C}$ called a coproduct and $\epsilon_\mathcal{C} : \mathcal{C} \to R$ called a counit, such that

$$(\Delta_\mathcal{C} \otimes_R \mathcal{C}) \circ \Delta_\mathcal{C} = (\mathcal{C} \otimes_R \Delta_\mathcal{C}) \circ \Delta_\mathcal{C}, \quad (\epsilon_\mathcal{C} \otimes_R \mathcal{C}) \circ \Delta_\mathcal{C} = (\mathcal{C} \otimes_R \epsilon_\mathcal{C}) \circ \Delta_\mathcal{C} = \mathcal{C}.$$

We use the Sweedler notation for the coproduct $\Delta_\mathcal{C}$ too. A left R-module M together with a left R-module map ${}^M\rho : M \to \mathcal{C} \otimes_R M$ such that

$$(\mathcal{C} \otimes_R {}^M\rho) \circ \rho = (\Delta_\mathcal{C} \otimes_R M) \circ {}^M\rho, \quad (\epsilon_\mathcal{C} \otimes_R M) \circ {}^M\rho = M$$

is called a *left comodule of the coring* $\mathcal{C}$ or, simply, a *left $\mathcal{C}$-comodule*, and ${}^M\rho$ is called a *left coaction*. A map between left $\mathcal{C}$-comodules is a left R-module map $f : M \to N$ such that ${}^N\rho \circ f = (\mathcal{C} \otimes_R f) \circ {}^M\rho$. The category of left $\mathcal{C}$-comodules is denoted by ${}^\mathcal{C}_R\mathcal{M}$.

2.2. **R-rings and bialgebroids.** Let R be a k-algebra. Recall from [18], [20] that an *R-ring* is a pair (U,i), where U is a k-algebra and $i : R \to U$ is an algebra map. If (U,i) is an R-ring then U is an (R,R)-bimodule with the structure provided by the map i, $r \cdot u \cdot r' := i(r)ui(r')$.

Let R be an algebra and $\bar{R} = R^{op}$ the opposite algebra, and let $R^e = R \otimes \bar{R}$ be the enveloping algebra of R. In case (H,i) is an R^e-ring, the map i is necessarily of the form $i = m_H \circ (s_H \otimes t_H)$, where $s_H : R \to H$, $t_H : \bar{R} \to H$ are algebra maps such that $s_H(a)t_H(\bar{b}) = t_H(\bar{b})s_H(a)$, for all $a \in R$, $\bar{b} \in \bar{R}$, and m_H is the product in H. In this case s_H is called the *source* map and t_H the *target* map. $(H, i = m_H \circ (s_H \otimes t_H))$ is denoted by (H, s_H, t_H).

Let (H, s_H, t_H) be an R^e-ring and (A, s_A) an R-ring. We view H as an R-bimodule, via $r \cdot h \cdot r' = s_H(r)t_H(r')h$, and A as an R-bimodule via s_A, and define [20]

$$H \times_R A = \{\sum_i h^i \otimes_R a^i \in H \otimes_R A \mid \forall r \in R, \ \sum_i h^i t_H(r) \otimes_R a^i = \sum_i h^i \otimes_R a^i s_A(r)\}.$$

$H \times_R A$ is an R-ring with product

$$(\sum_i h^i \otimes_R a^i)(\sum_j \tilde{h}^j \otimes_R \tilde{a}^j) = \sum_{i,j} h^i\tilde{h}^j \otimes_R a^i\tilde{a}^j,$$

the unit $1_H \otimes_R 1_A$ and the algebra map $R \to H \times_R A$, $a \mapsto s_H(a) \otimes_R 1_A$ (cf. [20]). Taking $(A, s_A) = (H, s_H)$ we can define $H \times_R H$ which is not only an R-ring but also an R^e-ring via $R \otimes \bar{R} \to H \times_R H$, $a \otimes_R \bar{b} \mapsto s_H(a) \otimes_R t_H(\bar{b})$.

Definition 2.1. Let (H, s_H, t_H) be an R^e-ring. We say that $(H, s_H, t_H, \Delta, \epsilon)$ is an *R-bialgebroid* iff (H, Δ, ϵ) is an R-coring such that $\mathrm{Im}(\Delta) \subseteq H \times_R H$ and the corestriction

of the coproduct $\Delta : H \to H \times_R H$ is an algebra map, $\epsilon(1_H) = 1_R$, and for all $g, h \in H$

$$\epsilon(gh) = \epsilon\Big(gs_H(\epsilon(h))\Big) = \epsilon\Big(gt_H(\epsilon(h))\Big). \tag{1}$$

It is shown in [7] that this is equivalent both to the definition of a bialgebroid in [14] and that of $\times_R$-bialgebra in [20].

Szlachányi [19] has reformulated the definition of bialgebroid in terms of monoidal categories and monoidal functors: if H is an R^e-ring, then we have the restriction of scalars functor $F: {}_H\mathcal{M} \to {}_R\mathcal{M}_R$. H is an R-bialgebroid if and only if there exists a monoidal structure on ${}_H\mathcal{M}$ such that F is a strict monoidal functor. If $(H, s_H, t_H, \Delta, \epsilon)$ is as in Definition 2.1, then the corresponding monoidal structure on ${}_H\mathcal{M}$ is given by

$$h \triangleright (m \otimes_R n) = h_{(1)}m \otimes_R h_{(2)}n \ ; \ h \triangleright a = \varepsilon(hs(a)) = \varepsilon(ht(a))$$

for all $m \in M \in {}_H\mathcal{M}$, $n \in N \in {}_H\mathcal{M}$, $a \in R$.

Basic examples of R-bialgebroids are provided by R^e and $\mathrm{End}(R)$, in the case R is finitely generated projective over k (see [14], [20]). In particular any matrix algebra $M_n(k)$ has a structure of an R-bialgebroid with an antipode over any n-dimensional algebra R. We believe, this gives a nice motivation for studying bialgebroids from an algebraic point of view.

2.3. **Doi-Koppinen datum over a weak Hopf algebra.** A *weak bialgebra* is an algebra and a coalgebra H with multiplicative (but non-unital) coproduct such that for all $x, y, z \in H$, $\epsilon(xyz) = \epsilon(xy_{(1)})\epsilon(y_{(2)}z) = \epsilon(xy_{(2)})\epsilon(y_{(1)}z)$, and

$$(\Delta \otimes H) \circ \Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1). \tag{2}$$

A *weak Hopf algebra* is a weak bialgebra H with an antipode, i.e., a linear map $S : A \to A$ such that for all $h \in H$, $h_{(1)}S(h_{(2)}) = \epsilon(1_{(1)}h)1_{(2)}$, $S(h_{(1)})h_{(2)} = 1_{(1)}\epsilon(h1_{(2)})$, and $S(h_{(1)})h_{(2)}S(h_{(3)}) = S(h)$. Weak Hopf algebras have been introduced in [2] [15] and studied in connection to integrable models and classification of subfactors of von Neumann algebras. Given a weak Hopf algebra H with bijective antipode, define the maps,

$$\Pi^L, \Pi^R : H \to H, \quad \Pi^L(g) = \epsilon(1_{(1)}g)1_{(2)}, \quad \Pi^R(g) = \epsilon(g1_{(2)})1_{(1)}.$$

Then [3] $R := \mathrm{Im}(\Pi^L)$ is a subalgebra of H, separable and Frobenius as a k-algebra with the separability idempotent $e = S(1_{(1)}) \otimes 1_{(2)} \in R \otimes R$ and the Frobenius pair (e, φ), where $\varphi := \epsilon|_R$. The fact that e is a separability idempotent means explicitly

$$\forall g \in H, \quad \Pi^L(g)S(1_{(1)}) \otimes 1_{(2)} = S(1_{(1)}) \otimes 1_{(2)}\Pi^L(g). \tag{3}$$

Numerous useful formulae for a weak Hopf algebra were proven in [3]. Although some of them, such as (3), were obtained using duality arguments valid only in the finite dimensional case, one can also prove them directly using the axioms of a weak Hopf algebra. The proofs are not always obvious, but quite straightforward once one becomes familiar with these axioms.

Finally, recall from [1] the following

Definition 2.2. A left-left *weak Doi-Koppinen datum* is a triple (H, A, C), where H is a weak Hopf algebra and

(1) $(A, {}^A\rho)$ is a left *weak H-comodule algebra*, i.e., A is an algebra and a left H-comodule such that ${}^A\rho(a){}^A\rho(b) = {}^A\rho(ab)$, and $(H \otimes {}^A\rho) \circ {}^A\rho(1) = \sum 1_{(1)} \otimes 1_{<-1>}1_{(2)} \otimes 1_{<0>}$, for all $a, b \in A$;

(2) C is a left *weak H-module coalgebra*, i.e., C is a coalgebra with counit ϵ_C and a left H-module such that $\Delta_C(h \cdot c) = \sum h_{(1)} \cdot c_{(1)} \otimes h_{(2)} \cdot c_{(2)}$, and $\epsilon_C(hg \cdot c) = \epsilon_H(hg_{(2)})\epsilon_C(g_{(1)} \cdot c)$ for all $c \in C$ and $h, g \in H$.

A (left-left) *weak Doi-Koppinen module* associated to a weak Doi-Koppinen datum (H, A, C) is a triple $(M, \cdot, {}^M\rho)$, where $(M, \cdot)$ is a left A-module, $(M, {}^M\rho)$ is a left C-comodule, and

$$ {}^M\rho(a \cdot m) = a_{<-1>} \cdot m_{<-1>} \otimes a_{<0>} \cdot m_{<0>}. $$

Note that here ${}^A\rho(a) = a_{<-1>} \otimes a_{<0>} \in H \otimes C$ and ${}^M\rho(m) = m_{<-1>} \otimes m_{<0>} \in C \otimes M$. The category of (left-left) weak Doi-Koppinen modules is denoted by ${}^C_A\mathcal{M}(H)$.

Morphisms between left weak H-comodule algebras (resp. left weak H-module coalgebras) are defined in the obvious way: they are k-linear maps that are H-colinear (resp. H-linear) algebra (resp. coalgebra) maps. Thus we can consider the categories of left weak H-comodule algebras, left weak H-module coalgebras and left-left weak Doi-Koppinen data over H. The latter is denoted by $\mathcal{WDK}(H)$.

3. Doi-Koppinen modules over algebras

In this section we define the notion of a Doi-Koppinen datum over a noncommutative algebra and we relate it to a weak Doi-Koppinen datum. Our definition is in part motivated by the following important observation [12, Proposition 2.3.1].

Proposition 3.1. *Let H be a weak Hopf algebra with coproduct Δ, counit ϵ, and bijective antipode S, and let $R = \mathrm{Im}(\Pi^L)$. Then H is an R-bialgebroid with the source and target s_H, $t_H : R \to H$ given by*

$$ s_H(\Pi^L(g)) = \Pi^L(g), \quad t_H(\Pi^L(g)) = S^{-1}(\Pi^L(g)) = \epsilon(1_{(2)}g)1_{(1)}, $$

and the comultiplication $\tilde{\Delta} : H \to H \otimes_R H$ and the counit $\tilde{\epsilon} : H \to R$ given by

$$ \tilde{\Delta}(h) = (can \circ \Delta)(h) = h_{(1)} \otimes_R h_{(2)}, \quad \tilde{\epsilon}(h) = \Pi^L(h) $$

for all $h \in H$, where $can : H \otimes H \to H \otimes_R H$ is the canonical projection.

Proof. For the details we refer to [12], we only remark that $\mathrm{Im}(\tilde{\Delta}) \subseteq H \times_R H$ can be established from the separability of R as follows. Apply $S^{-1} \otimes H$ to (3), for an arbitrary $h \in H$ write $h = h1_H$, and use that Δ is multiplicative to obtain

$$ h_{(1)} \otimes h_{(2)}\Pi^L(g) = h_{(1)}1_{(1)} \otimes h_{(2)}1_{(2)}\Pi^L(g) = h_{(1)}S^{-1}(\Pi^L(g)) \otimes h_{(2)}. \tag{4} $$

We also note that in [12] the conditions (1) are not required for an R-bialgebroid. However it can be easily seen that $\tilde{\epsilon}$ as defined above satisfies equations (1). □

There is also a partial converse to Proposition 3.1 (cf. [19, Proposition 1.6]). Suppose H is an R-bialgebroid, where R is a Frobenius and separable k-algebra. Let $e = e^{(1)} \otimes e^{(2)}$ (summation understood) be a separability idempotent and let $\varphi : R \to k$ be the Frobenius map such that (e, φ) is a Frobenius pair. Then H is a weak bialgebra with the coproduct $\tilde{\Delta} : H \to H \otimes H$, $h \mapsto h_{(1)} \cdot e^{(1)} \otimes e^{(2)} \cdot h_{(2)}$, and the counit $\tilde{\epsilon} = \varphi \circ \epsilon : H \to k$, where $\epsilon : H \to R$ is the counit of the R-coring H.

Definition 3.2. Let (H, s_H, t_H) be an R-bialgebroid. Then a *left H-module coalgebra* is a coalgebra C in the monoidal category $({}_H\mathcal{M}, \otimes_R, R)$ of left H-modules.

Recall from [19] that ${}_H\mathcal{M}$ has a monoidal structure defined as follows. For all $M, N \in {}_H\mathcal{M}$, $M \otimes_R N \in {}_H\mathcal{M}$ via $h \cdot (m \otimes_R n) = h_{(1)} \cdot m \otimes_R h_{(2)} \cdot n$. R is the unit object, when viewed in ${}_H\mathcal{M}$ via the action

$$ h \triangleright a = \epsilon(hs_H(a)) = \epsilon(ht_H(a)). \tag{5} $$

Thus, C is a left H-module coalgebra if and only if $(C, \cdot)$ is a left H-module and $(C, \Delta_C, \epsilon_C)$ is an R-coring, where C is viewed as an R-bimodule via $r \cdot c \cdot r' =$

$s_H(r)t_H(r')\cdot c$, such that Δ_C, ϵ_C are left H-modules maps, i.e., for all $h\in H$ and $c\in C$,

$$\Delta_C(h\cdot c)=h_{(1)}\cdot c_{(1)}\otimes_R h_{(2)}\cdot c_{(2)},\quad \epsilon_C(h\cdot c)=h\triangleright\epsilon_C(c)=\epsilon_H(hs_H(\epsilon_C(c))). \tag{6}$$

A morphism between two H-module coalgebras is an H-linear map of R-corings. We can then consider the category of H-module coalgebras.

Example 3.3. (1) (H,Δ_H,ϵ_H) is a left H-module coalgebra with the left H-action provided by the left multiplication.

(2) View R as an R-coring in the trivial way, i.e., both Δ_R and ϵ_R are identity maps. Then $(R,\triangleright)$ is a left H-module coalgebra. Indeed, note that for all $r\in R$ and $h\in H$ we have that $s_H(h_{(1)}\triangleright r)h_{(2)}=hs_H(r)$ [7], and then apply the left R-module map ϵ_H to obtain that $\epsilon_H(hs_H(r))=\epsilon_H(h_{(1)}s_H(r))\epsilon_H(h_{(2)})$. This is equivalent to the fact that Δ_R is a left H-linear map.

(3) $C=R^e$ is an R-coring with the coproduct $\Delta_{R^e}(r\otimes\bar r)=r\otimes 1_{\bar R}\otimes_R 1_R\otimes\bar r$ and the counit $\epsilon_{R^e}(r\otimes\bar r)=r\bar r$, and it can be made into a left H-module coalgebra by the H-action $h\cdot(r\otimes\bar r)=\epsilon_H(hs_H(r)t_H(\bar r))$.

Definition 3.4. Let (H,s_H,t_H) be an R-bialgebroid. A *left H-comodule algebra* is a triple $(A,s_A,{}^A\rho)$ where

(1) (A,s_A) is an R-ring.
(2) $(A,{}^A\rho)$ is a left comodule of the R-coring H.
(3) $\mathrm{Im}({}^A\rho)\subseteq H\times_R A$ and its corestriction ${}^A\rho:A\to H\times_R A$ is an algebra map.

A morphism between two H-comodule algebras is a left H-colinear map that is also a morphism of R-rings. A morphism of R-rings is defined in the obvious way. Thus we can consider the category of H-comodule algebras.

Example 3.5. (1) (H,s_H,Δ) is a left H-comodule algebra.

(2) $(R,s_R=R,{}^R\rho)$, where ${}^R\rho:R\to H\otimes_R R$, $r\mapsto s_H(r)\otimes_R 1_R$ is a (trivial) left H-comodule algebra

(3) $A=R^e$ is a left H-comodule algebra via $s_{R^e}=R\otimes 1_R$, and ${}^{R^e}\rho(r\otimes\bar r)=s_H(r)\otimes_R 1_R\otimes\bar r$.

(4) Example (3) can be generalised as follows. For an H-comodule algebra A and an algebra B, $A\otimes B$ is a left H-comodule algebra with the structures arising from those of A.

(5) An interesting nontrivial example of a comodule algebra of a bialgebroid is constructed in [16, Theorem 6.3, Lemma 6.7]. Let H be a Hopf algebra and let A/B be an H-Galois extension, i.e., A be a right H-comodule algebra with a right coaction $\rho^A:A\to A\otimes H$, $a\mapsto a_{<0>}\otimes a_{<1>}$, $B=A^{coH}$, and the canonical map $\chi:A\otimes_B A\to A\otimes H$, $a\otimes_B a'\mapsto aa'_{<0>}\otimes a'_{<1>}$ be bijective. Suppose H is k-flat and A is a faithfully flat left B-module. View A^e as a right H-comodule via $a\otimes\bar a\mapsto a_{<0>}\otimes\bar a_{<0>}\otimes a_{<1>}\bar a_{<1>}$. Then the k-module of coinvariants $G=(A^e)^{coH}$ is a subalgebra of A^e and a B-bialgebroid via $s_G:b\mapsto b\otimes 1$, $t_G:b\mapsto 1\otimes b$, $\Delta_G:\sum_i a^i\otimes\bar a^i\mapsto\sum_i a^i{}_{<0>}\otimes\chi^{-1}(1\otimes a^i{}_{<1>})\otimes\bar a^i$, and $\epsilon_G:\sum_i a^i\otimes\bar a^i\mapsto\sum_i a^i\bar a^i$. Furthermore A is a left comodule algebra of the B-bialgebroid G with a left coaction ${}^A\rho:A\to G\otimes_B A$, $a\mapsto a_{<0>}\otimes\chi^{-1}(1\otimes a_{<1>})$.

Note that the definition of a left H-comodule algebra is not dual to that of a left H-module coalgebra. The reason is that, although the category of left H-modules is monoidal, the category ${}^H_R\mathcal{M}$ of left comodules of an R-coring H is not. Thus there is no way of defining a left H-module algebra as an algebra in the category ${}^H_R\mathcal{M}$.

However one can consider more restrictive definition of a left H-comodule M (cf. [16, Definition 5.5]) by requiring it to be an R-bimodule with an R-bimodule coaction ${}^M\rho : M \to H \otimes_R M$ such that $\mathrm{Im}({}^M\rho) \subseteq H \times_R M$, where

$$H \times_R M = \{\sum_i h^i \otimes_R m^i \in H \otimes_R M \mid \forall r \in R,\ \sum_i h^i t_H(r) \otimes_R m^i = \sum_i h^i \otimes_R m^i \cdot r\}.$$

The subcategory ${}^H_R\mathcal{M}_R \subseteq {}^H_R\mathcal{M}$ of all such comodules is monoidal. For all M, $N \in {}^H_R\mathcal{M}_R$, $M \otimes_R N \in {}^H_R\mathcal{M}_R$ via ${}^{M\otimes_R N}\rho(m \otimes_R n) = m_{<-1>}n_{<-1>} \otimes_R m_{<0>} \otimes_R n_{<0>}$. Note that the right hand side is well defined because $\mathrm{Im}({}^M\rho) \subseteq H \times_R M$. R is the unit in ${}^H_R\mathcal{M}_R$ with the trivial coaction ${}^R\rho(r) = s_H(r) \otimes_R 1_R$. Furthermore the forgetful functor $F : {}^H_R\mathcal{M}_R \to {}_R\mathcal{M}_R$ is strict monoidal. Now, one could define a left H-comodule algebra as an algebra in the monoidal category $({}^H_R\mathcal{M}_R, \otimes_R, R)$. It appears, however, that this definition is too restrictive to cover the case of a weak Doi-Koppinen datum.

Definition 3.6. Let $(H, s_H, t_H, \Delta, \epsilon)$ be an R-bialgebroid. Then (H, A, C) is called a *(left-left) Doi-Koppinen datum over (an algebra) R* if A is a left H-comodule algebra and C is a left H-module coalgebra. Such a datum is denoted by $(H, A, C)_R$. The category of Doi-Koppinen data with H over R is denoted by $\mathcal{DK}_R(H)$.

A *(left-left) Doi-Koppinen module* over R (associated to $(H, A, C)_R$) is a triple $(M, \cdot, {}^M\rho)$, where $(M, \cdot)$ is a left A-module (hence M is a left R-module via s_A), $(M, {}^M\rho)$ is a left comodule of the R-coring C, and for all $a \in A$ and $m \in M$,

$$ {}^M\rho(a \cdot m) = a_{<-1>} \cdot m_{<-1>} \otimes_R a_{<0>} \cdot m_{<0>}. \tag{7}$$

The category of Doi-Koppinen modules associated to $(H, A, C)_R$ is denoted by ${}^C_A\mathcal{M}(H; R)$. Note that the right hand side of (7) is well defined since $\mathrm{Im}({}^A\rho) \subseteq H \times_R A$.

Example 3.7. There are various examples of special cases of the category ${}^C_A\mathcal{M}(H; R)$ obtained by setting $A = H, R, R^e$ and $C = H, R, R^e$. In particular, the category of left H-modules, the category of left H-comodules or the category of (generalised) relative Hopf modules ${}^H_A\mathcal{M}(H; R)$ and its dual ${}^C_H\mathcal{M}(H; R)$ are all special cases of the category ${}^C_A\mathcal{M}(H; R)$.

The main aim of this section is to show that a weak Doi-Koppinen datum in Definition 2.2 is a special case of a Doi-Koppinen datum over a noncommutative algebra. This provides one with a new point of view on weak Doi-Koppinen modules. In the proof of the next two Propositions, we will make use of the following remark.

Remark 3.8. If S is a separable k-algebra with an idempotent $e = e^{(1)} \otimes e^{(2)}$ (summation understood), and M and N are S-bimodules then the canonical projection $M \otimes N \to M \otimes_S N$ has a section

$$\sigma : M \otimes_S N \to M \otimes N, \quad \sigma(m \otimes_S n) = m \cdot e^{(1)} \otimes e^{(2)} \cdot n.$$

If H is a weak Hopf algebra, then H is a bialgebroid over a separable (and Frobenius) algebra R with idempotent $e = S(1_{(1)}) \otimes 1_{(2)}$ and the Frobenius map $\varphi : R \to k$, $\varphi := \epsilon|_R$, the restriction of a weak counit of H to R.

Proposition 3.9. *Let H be a weak Hopf algebra viewed as an R-bialgebroid as in Proposition 3.1. Then the category of left weak H-comodule algebras (in the sense of Definition 2.2) and of left comodule algebras over the bialgebroid H (in the sense of Definition 3.4) are isomorphic to each other.*

Proof. 1) Let A be a left H-comodule algebra. We will show that A is then a left comodule algebra over the R-bialgebroid H. Note that

$$s_A:\ R \to A, \quad s_A(\Pi^L(h)) = \epsilon(1_{<-1>}h)1_{<0>},$$

where $\epsilon: H \to k$ is a weak counit, is a well-defined algebra map. Indeed, suppose $r = \Pi^L(h) = 0$. This means that $\epsilon(1_{(1)}h)1_{(2)} = 0$, and therefore

$$s_A(r) = \epsilon(1_{<-2>}h)\epsilon(1_{<-1>})1_{<0>} = \epsilon(1_{(1)}h)\epsilon(1_{(2)}1_{<-1>})1_{<0>} = 0.$$

Note that the second equality was obtained by using the following observation made in [8, Proposition 4.11]. The unit property of a coaction of a weak comodule algebra in Definition 2.2(1) is equivalent to

$$1_{<-2>} \otimes 1_{<-1>} \otimes 1_{<0>} = 1_{(1)} \otimes 1_{(2)}1_{<-1>} \otimes 1_{<0>}. \tag{8}$$

This proves that s_A is well-defined. To prove that s_A is an algebra map we require the following two equalities (cf. [3, Eq. (2.9b)] and [1, Eq. (2.1b)] respectively). For all $g, h \in H$ and $a \in A$,

$$\Pi^R(g)h = h_{(1)}\epsilon(gh_{(2)}), \qquad \Pi^R(a_{<-1>}) \otimes a_{<0>} = 1_{<-1>} \otimes a1_{<0>}. \tag{9}$$

Now for all $r = \Pi^L(h)$, $s = \Pi^L(g)$ we have

$$\begin{aligned}
s_A(rs) &= s_A(\Pi^L(h)\Pi^L(g))\\
&= s_A\Big(\Pi^L(\Pi^L(h)g)\Big) = \epsilon\Big(1_{<-1>}\epsilon(1_{(1)}h)1_{(2)}g\Big)1_{<0>}\\
&= \epsilon(1_{<-2>}h)\epsilon(1_{<-1>}g)1_{<0>} = \epsilon(1_{<-2>}h)\epsilon(g_{(1)})\epsilon(1_{<-1>}g_{(2)})1_{<0>}\\
\text{(by Eqs (9))} \quad &= \epsilon(1_{<-2>}h)\epsilon(\Pi^R(1_{<-1>})g)1_{<0>} = \epsilon(1_{<-1>}h)\epsilon(1_{<-1'>}g)1_{<0>}1_{<0'>}\\
&= s_A(r)s_A(s),
\end{aligned}$$

where $1_{<-1'>} \otimes 1_{<0'>}$ denotes another copy of $1_{<-1>} \otimes 1_{<0>}$, and [3, Lemma 2.5] has been used to derive the second equality and the unit property in Definition 2.2 (1) to obtain the fourth one. This proves that s_A is an algebra map and hence (A, s_A) is an R-ring.

Next, using the canonical projection $can: H \otimes A \to H \otimes_R A$ define a map $\tilde{\rho} = can \circ {}^A\rho$, $\tilde{\rho}: A \to H \otimes_R A$, where ${}^A\rho: A \to H \otimes A$ is the left coaction of a weak Hopf algebra. Explicitly, $\tilde{\rho}(a) = a_{<-1>} \otimes_R a_{<0>}$. The map $\tilde{\rho}$ is left R-linear since using equation (8) and the fact that A is a comodule algebra we have for all $r = \Pi^L(h) \in R$ and $a \in A$

$$\begin{aligned}
\tilde{\rho}(r \cdot a) &= \tilde{\rho}(\epsilon(1_{<-1>}h)1_{<0>}a) = \epsilon(1_{<-2>}h)1_{<-1>}a_{<-1>} \otimes_R 1_{<0>}a_{<0>}\\
&= \epsilon(1_{(1)}h)1_{(2)}1_{<-1>}a_{<-1>} \otimes_R 1_{<0>}a_{<0>} = \Pi^L(h)a_{<-1>} \otimes_R a_{<0>}.
\end{aligned}$$

Then it is clear that $(A, \tilde{\rho}) \in {}^H_R\mathcal{M}$. We prove now that $\mathrm{Im}(\tilde{\rho}) \subseteq H \times_R A$. Equation (4) implies that for all $g \in H$ and $a \in A$

$$a_{<-2>} \otimes a_{<-1>}\Pi^L(g) \otimes a_{<0>} = a_{<-2>}S^{-1}(\Pi^L(g)) \otimes a_{<-1>} \otimes a_{<0>}.$$

Apply $H \otimes \epsilon \otimes A$ to the last equality to obtain

$$a_{<-2>} \otimes \epsilon(a_{<-1>}\Pi^L(g))a_{<0>} = a_{<-1>}S^{-1}(\Pi^L(g)) \otimes a_{<0>}$$

Using equations (2.2a),(2.2b) in [3] which, put together, state that for all $g, h \in H$, $\epsilon(g\Pi^L(h)) = \epsilon(\Pi^R(g)h)$ we compute

$$\begin{aligned}
a_{<-2>} \otimes \epsilon(a_{<-1>}\Pi^L(g))a_{<0>} &= a_{<-2>} \otimes \epsilon(\Pi^R(a_{<-1>})g)a_{<0>}\\
\text{(by Eq. (9))} \quad &= a_{<-1>} \otimes \epsilon(1_{<-1>}g)a_{<0>}1_{<0>}\\
&= a_{<-1>} \otimes a_{<0>}s_A(\Pi^L(g)).
\end{aligned}$$

Hence we have proved that

$$a_{<-1>}S^{-1}(\Pi^L(g))\otimes a_{<0>} = a_{<-1>}\otimes a_{<0>}s_A(\Pi^L(g)). \tag{10}$$

In particular, $\mathrm{Im}(\tilde{\rho})\subseteq H\times_R A$ as required. It remains to be proven that $\tilde{\rho}(1_A) = 1_H\otimes_R 1_A$. Using the unit property of a comodule algebra of a weak Hopf algebra in Definition 2.2(1) we compute

$$\rho(1_A) = 1_{<-2>}\otimes\epsilon(1_{<-1>})1_{<0>} = 1_{(1)}\otimes\epsilon(1_{<-1>}1_{(2)})1_{<0>} = 1_{(1)}\otimes s_A(1_{(2)}).$$

Since A is a left R-module via s_A we obtain

$$1_{<-1>}\otimes_R 1_{<0>} = 1_{(1)}\otimes_R s_A(1_{(2)}) = 1_{(1)}\cdot 1_{(2)}\otimes_R 1_A = S^{-1}(1_{(2)})1_{(1)}\otimes_R 1_A = 1_H\otimes_R 1_A.$$

This completes the proof that $(A, s_A, \tilde{\rho})$ is a left H-comodule algebra over the R-bialgebroid H.

If $f: (A, {}^A\rho)\to(B, {}^B\rho)$ is a morphism of left H-comodule algebras, then $f: (A, s_A, \widetilde{\rho})\to(B, s_B, \widetilde{\rho})$ is also a morphism of left H-comodule algebras over the R-bialgebroid H. In view of the definition of $\widetilde{\rho}$, the left H-colinearity is obvious. We also know that f is an algebra map, so f is a map of R-rings if $s_B = f\circ s_A$. Using the H-colinearity of f and the fact that $f(1_A) = 1_B$, we find

$$\begin{aligned} f(s_A(\pi^L(h)) &= \epsilon(1_{A<-1>}h)f(1_{A<0>}) = \epsilon(f(1_A)_{<-1>}h)f(1_A)_{<0>}\\ &= \epsilon(1_{B<-1>}h)f(1_{B<0>}) = s_B(\pi^L(h)) \end{aligned}$$

2) Conversely, let $(A, s_A, {}^A\rho)$ be a left comodule algebra over the bialgebroid H as in Definition 3.4. We prove that A is a weak left H-comodule algebra with coaction given by $\tilde{\rho} = \sigma\circ{}^A\rho$. Explicitly

$$\tilde{\rho}(a) = a_{<-1>}\cdot S(1_{(1)})\otimes 1_{(2)}\cdot a_{<0>} = 1_{(1)}a_{<-1>}\otimes 1_{(2)}\cdot a_{<0>} = 1_{(1)}a_{<-1>}\otimes s_A(1_{(2)})a_{<0>},$$

where we used that H is a right R-module via the target map $t_H = S^{-1}\mid_R$. The fact that $(A, \tilde{\rho})$ is a left comodule of a weak Hopf algebra H can easily be established with the help of equations (2). We prove now that $\tilde{\rho}$ is an algebra map. First, since $\mathrm{Im}({}^A\rho)\subseteq H\times_R A$ we have for all $r\in R$ and $a\in A$,

$$a_{<-1>}t_H(r)\otimes_R a_{<0>} = a_{<-1>}\otimes_R a_{<0>}s_A(r).$$

Applying the section σ we obtain

$$1_{(1)}a_{<-1>}t_H(r)\otimes s_A(1_{(2)})a_{<0>} = 1_{(1)}a_{<-1>}\otimes s_A(1_{(2)})a_{<0>}s_A(r). \tag{11}$$

On the other hand, application of σ to an expression reflecting the fact that ${}^A\rho : A\to H\times_R A$ is an algebra map leads to equality

$$1_{(1)}(ab)_{<-1>}\otimes s_A(1_{(2)})(ab)_{<0>} = 1_{(1)}a_{<-1>}b_{<-1>}\otimes s_A(1_{(2)})a_{<0>}b_{<0>} \tag{12}$$

Noting that $t_H = S^{-1}\mid_R$ and writing $1_{(1')}\otimes 1_{(2')}$ for another copy of $1_{(1)}\otimes 1_{(2)}$ we have

$$\begin{aligned} \tilde{\rho}(a)\tilde{\rho}(b) &= 1_{(1)}a_{<-1>}1_{(1')}b_{<-1>}\otimes s_A(1_{(2)})a_{<0>}s_A(1_{(2')})b_{<0>}\\ \text{(by Eq. (11))}\quad &= 1_{(1)}a_{<-1>}S^{-1}(1_{(2')})1_{(1')}b_{<-1>}\otimes s_A(1_{(2)})a_{<0>}b_{<0>}\\ \text{(by Eq. (12))}\quad &= 1_{(1)}(ab)_{<-1>}\otimes s_A(1_{(2)})(ab)_{<0>} = \tilde{\rho}(ab), \end{aligned}$$

i.e., $\tilde{\rho}$ is multiplicative as required. It remains to prove the unit property of the weak coaction $\tilde{\rho}$. Since $\tilde{\rho}(1) = 1_{(1)}1_{<-1>}\otimes 1_{(2)}\cdot 1_{<0>}$ we obtain that

$$(H\otimes\tilde{\rho})\circ\tilde{\rho}(1) = 1_{(1)}1_{<-2>}\otimes 1_{(2)}1_{<-1>}\otimes 1_{(3)}\cdot 1_{<0>},$$

hence the unit property of $\tilde{\rho}$ is equivalent to the following equation

$$1_{(1')}\otimes 1_{(1)}1_{<-1>}1_{(2')}\otimes 1_{(2)}\cdot 1_{<0>} = 1_{(1)}1_{<-2>}\otimes 1_{(2)}1_{<-1>}\otimes 1_{(3)}\cdot 1_{<0>}.$$

It is known, however, that

$$1_{<-1>} \otimes_R 1_{<0>} = 1_H \otimes_R 1_A \tag{13}$$

for $(A, {}^A\rho)$ is a comodule algebra of an R-bialgebroid. Application of f yields

$$1_{(1)} 1_{<-1>} \otimes 1_{(2)} \cdot 1_{<0>} = 1_{(1)} \otimes s_A(1_{(2)}). \tag{14}$$

Next apply $\Delta \otimes A$ to the preceding equality to obtain

$$1_{(1)} 1_{<-2>} \otimes 1_{(2)} 1_{<-1>} \otimes 1_{(3)} \cdot 1_{<0>} = 1_{(1)} \otimes 1_{(2)} \otimes s_A(1_{(3)}).$$

Now the required condition follows from these two equations, and equation (2). Thus we conclude that $(A, \tilde{\rho})$ is a left weak H-comodule algebra.

Suppose now that $f : (A, s_A, {}^A\rho) \to (B, s_B, {}^B\rho)$ is a morphism of left H-comodule algebras over the R-bialgebroid H. Then f is an algebra map, and

$$a_{<-1>} \otimes_R f(a_{<0>}) = f(a)_{<-1>} \otimes_R f(a)_{<0>}.$$

Applying σ to both sides, we obtain

$$a_{<-1>} \cdot S(1_{(1)}) \otimes 1_{(2)} \cdot f(a_{<0>}) = f(a)_{<-1>} \cdot S(1_{(1)}) \otimes 1_{(2)} \cdot f(a)_{<0>}.$$

f is a map of left R-modules, so the left hand side equals $a_{<-1>} \cdot S(1_{(1)}) \otimes f(1_{(2)} \cdot a_{<0>})$, and this means that $f : (A, \widetilde{\rho}) \to (B, \widetilde{\rho})$ is left H-colinear, and f is a morphism of left H-comodule algebras.

3) We still need to show that the functors constructed in parts 1) and 2) of the proof are inverses to each other. First, let (A, ρ) be a left weak H-comodule algebra. It is first transformed into a left H-comodule algebra $(A, \widetilde{\rho}, s_A)$ over the bialgebroid R, and then into a left weak H-comodule algebra $(A, \overline{\rho})$. We easily compute that

$$\begin{aligned}
\overline{\rho}(a) &= a_{<-1>} \cdot S(1_{(1)}) \otimes 1_{(2)} \cdot a_{<0>} = 1_{(1)} a_{<-1>} \otimes s_A(1_{(2)}) a_{<0>} \\
&= 1_{(1)} a_{<-1>} \otimes \epsilon(1_{<-1>} 1_{(2)}) 1_{<0>} a_{<0>} = 1_{<-2>} a_{<-1>} \otimes \epsilon(1_{<-1>}) 1_{<0>} a_{<0>} \\
&= 1_{<-1>} a_{<-1>} \otimes 1_{<0>} a_{<0>} = a_{<-1>} \otimes a_{<0>} = \rho(a),
\end{aligned}$$

as needed. We used Definition 2.2 (1).

Conversely, we start with a left H-comodule algebra (A, ρ, s_A) over the bialgebroid R, transform it into a weak left H-comodule algebra $(A, \widetilde{\rho})$ using part 2), and then back into $(A, \overline{\rho}, \overline{s}_A)$ over R. We have to show that $\rho = \overline{\rho}$ and $s_A = \overline{s}_A$. We write $\rho(a) = a_{<-1>} \otimes_R a_{<0>}$, and then easily find that

$$\begin{aligned}
\overline{\rho}(a) &= a_{<-1>} \cdot S(1_{(1)}) \otimes_R 1_{(2)} \cdot a_{<0>} \\
&= a_{<-1>} \cdot (S(1_{(1)}) 1_{(2)}) \otimes_R a_{<0>} = a_{<-1>} \otimes_R a_{<0>} = \rho(a).
\end{aligned}$$

Using equation (14), we obtain

$$\overline{s}_A(\Pi^L(h)) = \epsilon(1_{(1)} 1_{<-1>} h) s_A(1_{(2)}) 1_{<0>} = s_A(\epsilon(1_{(1)} h) 1_{(2)}) = s_A(\Pi^L(h)).$$

□

Proposition 3.10. *Let H be a weak Hopf algebra viewed as an R-bialgebroid as in Proposition 3.1. The categories of left weak H-module coalgebras (in the sense of Definition 2.2) and of left module coalgebras over the bialgebroid H (in the sense of Definition 3.2) are isomorphic to each other.*

Proof. 1) Let $(C, \Delta_C, \epsilon_C)$ be a left weak H-module coalgebra and define $\tilde{\Delta}_C : C \to C \otimes_R C$, $c \mapsto c_{(1)} \otimes_R c_{(2)}$ and $\tilde{\epsilon}_C : C \to R$, $c \mapsto \epsilon_C(1_{(1)} \cdot c)1_{(2)}$. We will show that these maps make C into a left module coalgebra of the R-bialgebroid H.

Since Δ_C is left H-linear, so is $\tilde{\Delta}_C$. This implies that $\tilde{\Delta}_C$ is an R-bimodule map. Next we prove that $\tilde{\epsilon}_C$ is left H-linear. First note that setting $g = 1_H$ in Definition 2.2 (2) one immediately obtains $\epsilon_C(h \cdot c) = \epsilon_H(h1_{(2)})\epsilon_C(1_{(1)} \cdot c)$ for all $h \in H$, $c \in C$. In particular

$$\begin{aligned}\tilde{\epsilon}_C(h \cdot c) &= \epsilon_C(1_{(1')} \cdot c)\epsilon_H(1_{(1)}h1_{(2')})1_{(2)} = \Pi^L(h1_{(2)})\epsilon_C(1_{(1)} \cdot c)\\ &= \tilde{\epsilon}_H(h1_{(2)})\epsilon_C(1_{(1)} \cdot c) = \tilde{\epsilon}_H\Big(h\Pi^L(\tilde{\epsilon}_C(c))\Big) = h \triangleright \tilde{\epsilon}_C(c),\end{aligned}$$

where $1_{(1')} \otimes 1_{(2')}$ is another copy of $\Delta(1_H)$ and $\tilde{\epsilon}_H : H \to R$ is the counit of H as an R-bialgebroid. This implies that $\tilde{\epsilon}_C$ is an R-bimodule map. Clearly, $\tilde{\Delta}_C$ is a coproduct and $\tilde{\epsilon}_C$ is a counit of an R-coring C. The compatibility of $\tilde{\Delta}_C$ with the left action of H on C follows immediately from the fact that C is a weak module coalgebra. Thus we conclude that C is a left module coalgebra over the R-bialgebroid H as claimed.

Let $f : (C, \Delta_C, \epsilon_C) \to (D, \Delta_D, \epsilon_D)$ be a morphism of left weak H-module coalgebras. Then f is left H-linear, and it clearly preserves the comultiplication over R. Furthermore

$$\widetilde{\epsilon}_D(f(c)) = \epsilon_D(1_{(1)} \cdot f(c))1_{(2)} = \epsilon_D(f(1_{(1)} \cdot c))1_{(2)} = \epsilon_C(1_{(1)} \cdot c)1_{(2)} = \widetilde{\epsilon}_C(c)$$

and we conclude that $f : (C, \widetilde{\Delta}_C, \widetilde{\epsilon}_C) \to (D, \widetilde{\Delta}_D, \widetilde{\epsilon}_D)$ is a morphism of left H-module coalgebras over the R-bialgebroid H.

2) Conversely, assume that C is a left module coalgebra over the R-bialgebroid H, in the sense of Definition 3.2. We claim that C is a left weak H-module coalgebra with the coproduct

$$\tilde{\Delta}_C : C \to C \otimes C, \quad \tilde{\Delta}_C(c) = c_{(1)} \cdot S(1_{(1)}) \otimes 1_{(2)} \cdot c_{(2)} = 1_{(1)} \cdot c_{(1)} \otimes 1_{(2)} \cdot c_{(2)}$$

(note that here $1_{(1)} \otimes 1_{(2)} = \Delta(1_H)$ while $c_{(1)} \otimes_R c_{(2)} = \Delta_C(c)$), and the counit $\tilde{\epsilon}_C : C \to k$, $c \mapsto \varphi(\epsilon_C(c))$, where $\epsilon_C : C \to R$ is the counit of the R-coring C and $\varphi : R \to k$ is the Frobenius map $\varphi := \epsilon|_R$, the restriction of the counit of a weak Hopf algebra H to the Frobenius algebra R. This claim can be proven by a fairly straightforward calculation and hence details of the proof are left to the reader.

Let $f : (C, \Delta_C, \epsilon_C) \to (D, \Delta_D, \epsilon_D)$ be a morphism of left H-module coalgebras over the bialgebroid H. Then f is left H-linear and

$$(f \otimes f)\widetilde{\Delta}_C(c) = f(c_{(1)} \cdot S(1_{(1)})) \otimes f(1_{(2)} \cdot c_{(2)}) = f(c_{(1)}) \cdot S(1_{(1)}) \otimes 1_{(2)} \cdot f(c_{(2)}) = \widetilde{\Delta}_D(f(c))$$

and

$$\widetilde{\epsilon}_D(f(c)) = \varphi(\epsilon_D(f(c))) = \varphi(\epsilon_C(c)) = \widetilde{\epsilon}_C(c),$$

so that f is also a morphism in the category of weak left H-module coalgebras.

3) We finally prove that the functors constructed in parts 1 and 2 of the proof are inverses to each other. First we take a left weak H-module coalgebra $(C, \Delta_C, \epsilon_C)$, turn it into a left module coalgebra $(C, \widetilde{\Delta}_C, \widetilde{\epsilon}_C)$ over the bialgebroid H, and then back into a left weak H-module coalgebra $(C, \overline{\Delta}_C, \overline{\epsilon}_C)$. Using Remark 3.8, we find

$$\overline{\Delta}_C(c) = c_{(1)} \cdot S(1_{(1)}) \otimes 1_{(2)} \cdot c_{(2)} = (\sigma \circ \mathrm{can})(c_{(1)} \otimes c_{(2)}) = \Delta_C(c)$$

and

$$\overline{\epsilon}_C(c) = \epsilon(\widetilde{\epsilon}_C(c)) = \epsilon_C(1_{(1)} \cdot c)\epsilon(1_{(2)}) = \epsilon_C(c).$$

Finally take a left module coalgebra $(C, \Delta_C, \epsilon_C)$ over the bialgebroid H, make it into a weak left H-module coalgebra $(C, \widetilde{\Delta}_C, \widetilde{\epsilon}_C)$, and then back into a left module coalgebra $(C, \overline{\Delta}_C, \overline{\epsilon}_C)$ over the bialgebroid H. Obviously

$$\overline{\Delta}_C(c) = c_{(1)} \cdot S(1_{(1)}) \otimes_R 1_{(2)} \cdot c_{(2)} = \Delta_C(c).$$

Proving $\overline{\epsilon} = \epsilon$ is slightly more complicated. Recall that $\epsilon : \ H \to k$ is the counit of the weak Hopf algebra H, and $\epsilon_H = \Pi^L : \ H \to R$ is the counit of the R-bialgebroid H. Take $c \in C$, and write $\epsilon_C(c) = \Pi^L(g) \in R$, for some $g \in H$. Observe that $\epsilon(\Pi^L(g)) = \epsilon(g)$ and $\widetilde{\epsilon}_C(c) = \epsilon(\epsilon_C(c))$, and compute

$$\begin{aligned}
\overline{\epsilon}_C(c) &= \widetilde{\epsilon}_C(1_{(1)} \cdot c)1_{(2)} = \epsilon(\epsilon_C(1_{(1)} \cdot c))1_{(2)} \\
\text{(by Eq. (6))} \quad &= \epsilon(1_{(1)} \triangleright \epsilon_C(c))1_{(2)} \\
\text{(by Eq. (5))} \quad &= \epsilon(\epsilon_H(1_{(1)} s_H(\epsilon_C(c))))1_{(2)} \\
&= \epsilon(\Pi^L(1_{(1)}\epsilon_C(c)))1_{(2)} = \epsilon(1_{(1)}\epsilon_C(c))1_{(2)} \\
&= \epsilon(1_{(1)}\Pi^L(g))1_{(2)} = \epsilon(1_{(1)}\epsilon(1'_{(1)}g)1'_{(2)})1_{(2)} \\
\text{(by Eq. (2))} \quad &= \epsilon(\epsilon(1_{(1)}g)1_{(2)})1_{(3)} \\
&= \epsilon(1_{(1)}g)1_{(2)} = \Pi^L(g) = \epsilon_C(c)
\end{aligned}$$

this completes the proof. □

Combining Propositions 3.9 and 3.10, we immediately obtain the following theorem, which is the main result of this Section. We leave it to the reader to define morphisms between left-left weak Doi-Koppinen data, and between left-left Doi-Koppinen data over an algebra R.

Theorem 3.11. *Let H be a weak Hopf algebra, and view it also as an R-bialgebroid, as in Proposition 3.1. Then there is an isomorphism of categories $\mathcal{WDK}(H) \cong \mathcal{DK}_R(H)$. Furthermore, the corresponding categories of Doi-Koppinen modules are isomorphic.*

4. A coring associated to a Doi-Koppinen datum over R and applications

In this section we construct an A-coring corresponding to a given Doi-Koppinen datum $(H, A, C)_R$ over R. This allows one to use methods employed in [5] to derive various properties of Doi-Koppinen modules over an algebra.

Proposition 4.1. *Let $(H, A, C)_R$ be a Doi-Koppinen datum over R. Then $\mathcal{C} = C \otimes_R A$ is an A-bimodule with the right action given by the multiplication in A and the left action $a \cdot (c \otimes_R a') = a_{<-1>} \cdot c \otimes_R a_{<0>}a'$, for all $a, a' \in A$, $c \in C$. Furthermore $\mathcal{C}$ is an A-coring with comultiplication $\Delta_{\mathcal{C}} = \Delta_C \otimes_R A$ and the counit $\epsilon_{\mathcal{C}} = \epsilon_C \otimes_R A$, where Δ_C, ϵ_C are the coproduct and the counit of the R-coring C. In this case the categories of left $\mathcal{C}$-comodules and of left-left Doi-Koppinen modules over R are isomorphic to each other.*

Proof. First note that the left action of A on $\mathcal{C}$ is well-defined since the image of the left H-coaction of A is required to be in $A \times_R H$. The fact that it is an action indeed follows from the fact that A is a left H-comodule algebra. Note also that in the definitions of $\Delta_{\mathcal{C}}$ and $\epsilon_{\mathcal{C}}$ we used the natural isomorphisms $C \otimes_R A \otimes_A C \otimes_R A \cong C \otimes_R C \otimes_R A$ and $R \otimes_R A \cong A$ respectively. Clearly $\Delta_{\mathcal{C}}$ is a right A-module map. To prove that it is a left A-module map as well take any $a, a' \in A$ and $c \in C$ and compute

$$\begin{aligned}
\Delta_{\mathcal{C}}(a \cdot (c \otimes_R a')) &= (a_{<-1>} \cdot c)_{(1)} \otimes_R (a_{<-1>} \cdot c)_{(2)} \otimes_R a_{<0>}a' \\
&= a_{<-2>} \cdot c_{(1)} \otimes_R a_{<-1>} \cdot c_{(2)} \otimes_R a_{<0>}a',
\end{aligned}$$

where we used that C is a left H-module coalgebra. On the other hand

$$\begin{aligned} a\cdot\Delta_{\mathcal{C}}(c\otimes_R a') &= a\cdot(c_{(1)}\otimes_R 1)\otimes_A(c_{(2)}\otimes_R a') = a_{<-1>}\cdot c_{(1)}\otimes_R a_{<0>}\cdot(c_{(2)}\otimes_R a') \\ &= a_{<-2>}\cdot c_{(1)}\otimes_R a_{<-1>}\cdot c_{(2)}\otimes_R a_{<0>}a'. \end{aligned}$$

This proves that $\Delta_{\mathcal{C}}$ is right A-linear, hence it is an A-bimodule map as required. Directly from the definition of $\Delta_{\mathcal{C}}$ it follows that it is coassociative.

It is clear that $\epsilon_{\mathcal{C}}$ is right A-linear. To prove that it is also a left A-module morphism take any $a, a'\in A$, $c\in C$ and compute

$$\begin{aligned} \epsilon_{\mathcal{C}}(a\cdot(c\otimes_R a')) &= \epsilon_C(a_{<-1>}\cdot c)\cdot(a_{<0>}a') = \epsilon_H(a_{<-1>}s_H(\epsilon_C(c)))\cdot(a_{<0>}a') \\ &= \epsilon_H(a_{<-1>}t_H(\epsilon_C(c)))\cdot(a_{<0>}a') = \epsilon_H(a_{<-1>})\cdot(a_{<0>}s_A(\epsilon_C(c))a') \\ &= a_{<0>}s_A(\epsilon_C(c))a' = a\epsilon_{\mathcal{C}}(c\otimes_R a'), \end{aligned}$$

where we used that C is a left H-module coalgebra to obtain the second equality, then equation (1) to derive the third one and the fact that the image of the left coaction of H on A is in $H\times_R A$ to obtain the fourth equality. This proves that $\epsilon_{\mathcal{C}}$ is left A-linear hence it is A-bilinear. The fact that $\epsilon_{\mathcal{C}}$ is a counit of $\mathcal{C}$ follows directly from the definition of $\epsilon_{\mathcal{C}}$. Thus we conclude that $\mathcal{C}$ is an A-coring as stated.

To prove the isomorphism of categories, take any left $\mathcal{C}$-comodule (M,ρ) and view it as a Doi-Koppinen module via the same coaction $\rho: M\to C\otimes_R A\otimes_A M\cong C\otimes_R M$. Conversely, any Doi-Koppinen module $(M,\cdot,\rho)$ can be viewed as a left $\mathcal{C}$ comodule via $\rho: M\to C\otimes_R M\cong C\otimes_R A\otimes_A M = \mathcal{C}\otimes_A M$. $\square$

One can now use the general results about corings in [5][1] combined with Proposition 4.1 to derive various properties of Doi-Koppinen modules over a noncommutative algebra R. For example [5, Lemma 3.1] implies that the forgetful functor $F: {}^C_A\mathcal{M}(H;R)\to {}_A\mathcal{M}$ is the left adjoint of the induction functor $G = C\otimes_R - : {}_A\mathcal{M}\to {}^C_A\mathcal{M}(H;R)$. Furthermore by [5, Theorems 3.3, 3.5] one has

Corollary 4.2. *Let $(H,A,C)_R$ be a left-left Doi-Koppinen datum over R.*

(1) The induction functor $G = C\otimes_R - : {}_A\mathcal{M}\to {}^C_A\mathcal{M}(H;R)$ is separable if and only if there exists $e=\sum_i c^i\otimes_R a^i\in C\otimes_R A$ such that $\sum_i \epsilon_C(c^i)\cdot a^i = 1_A$ and for all $a\in A$, $\sum_i a_{<-1>}\cdot c^i\otimes_R a_{<0>}a^i = \sum_i c^i\otimes_R a^i a$.

(2) The forgetful functor $F: {}^C_A\mathcal{M}(H;R)\to {}_A\mathcal{M}$ is separable if and only if there exists a right R-bimodule map $\gamma: C\otimes_R C\to A$ such that for all $a\in A$ and $c, c'\in C$,

- $\gamma(c_{(1)}\otimes_R c_{(2)}) = \epsilon_C(c)\cdot 1_A$,
- $\gamma(a_{<-2>}\cdot c\otimes_R a_{<-1>}\cdot c')a_{<0>} = a\gamma(c\otimes_R c')$,
- $c_{(1)}\otimes_R\gamma(c_{(2)}\otimes_R c') = \gamma(c\otimes_R c'_{(1)})_{<-1>}\cdot c'_{(2)}\otimes_R\gamma(c\otimes_R c'_{(1)})_{<0>}$.

Finally, it has been observed in [5, Proposition 2.3], that given a weak entwining structure (A,C,ψ), and hence a weak Doi-Koppinen datum in particular, one can construct a coring obtained as an image of certain projection in $C\otimes A$. Since a weak Doi-Koppinen datum is a special case of a Doi-Koppinen datum over R it is important to study the relationship of this coring to $C\otimes_R A$.

Proposition 4.3. *Let (H,A,C) be weak Doi-Koppinen datum. Define the corresponding A-coring $\tilde{\mathcal{C}} = \{\sum_i 1_{<-1>}\cdot c^i\otimes 1_{<0>}a^i \mid a^i\in A, c^i\in C\}$, with the coproduct $\Delta_{\tilde{\mathcal{C}}} = (\Delta_C\otimes A)\mid_{\tilde{\mathcal{C}}}$ and the counit $\epsilon_{\tilde{\mathcal{C}}} = (\epsilon_C\otimes A)\mid_{\tilde{\mathcal{C}}}$. View (H,A,C) as a Doi-Koppinen datum over $R=\mathrm{Im}\Pi^L$ in Theorem 3.11. Then $\tilde{\mathcal{C}}\cong\mathcal{C} = C\otimes_R A$ as A-corings.*

---

[1] Note, however, that some care has to be taken when applying [5] since this paper is formulated in the right-right module convention.

Proof. Consider two maps $\theta : \mathcal{C} \to \tilde{\mathcal{C}}$, $c \otimes_R a \mapsto 1_{<-1>} \cdot c \otimes 1_{<0>} a$, and $\tilde{\theta} : \tilde{\mathcal{C}} \to \mathcal{C}$, $\sum_i (c^i \otimes a^i) \mapsto \sum_i c^i \otimes_R a^i$. The map θ is well-defined because using equation (10) we have for all $a \in A$, $c \in C$ and $r \in R = \mathrm{Im}\Pi^L$,

$$\begin{aligned}\theta(c \cdot r \otimes_R a) &= 1_{<-1>} \cdot (c \cdot r) \otimes 1_{<0>} a = 1_{<-1>} t_H(r) \cdot c \otimes 1_{<0>} a \\ &= 1_{<-1>} \cdot c \otimes 1_{<0>} s_A(r) a = \theta(c \otimes_R r \cdot a).\end{aligned}$$

Clearly, θ is a right A-module map. Recall from [5, Proposition 2.3] that $\tilde{\mathcal{C}}$ is a left A-module via $a \cdot (\sum_i 1_{<-1>} \cdot c^i \otimes 1_{<0>} a^i) = \sum_i a_{<-1>} \cdot c^i \otimes a_{<0>} a^i$. Now, the fact that A is a weak left H-comodule algebra implies that θ is a left A-module map. Thus θ is an A-bimodule map. To prove that θ is a coring map take any $a \in A$, $c \in C$ and use the fact that A is an H-comodule algebra to compute

$$\begin{aligned}(\theta \otimes_A \theta) \circ \Delta_{\mathcal{C}}(c \otimes_R a) &= 1_{<-2>} \cdot c_{(1)} \otimes (1_{<-1>} 1_{<-1'>}) \cdot c_{(2)} \otimes 1_{<0>} 1_{<0'>} a \\ &= 1_{<-2>} \cdot c_{(1)} \otimes 1_{<-1>} \cdot c_{(2)} \otimes 1_{<0>} a,\end{aligned}$$

where $1_{<-1'>} \otimes 1_{<0'>}$ is another copy of $1_{<-1>} \otimes 1_{<0>}$. On the other hand, using the fact that C is a left H-module coalgebra we have

$$\Delta_{\tilde{\mathcal{C}}}(\theta(c \otimes_R a)) = (1_{<-1>} \cdot c)_{(1)} \otimes (1_{<-1>} \cdot c)_{(2)} \otimes 1_{<0>} a = 1_{<-2>} \cdot c_{(1)} \otimes 1_{<-1>} \cdot c_{(2)} \otimes 1_{<0>} a,$$

as required. Thus we have proven that θ is a map of A-corings. We now prove that $\tilde{\theta}$ is an inverse of θ. For a typical element $x = \sum_i 1_{<-1>} \cdot c^i \otimes 1_{<0>} a^i$ of $\tilde{\mathcal{C}}$ we have

$$\theta \circ \tilde{\theta}(x) = \sum_i 1_{<-1'>} 1_{<-1>} \cdot c^i \otimes 1_{<0'>} 1_{<0>} a^i = \sum_i 1_{<-1>} \cdot c^i \otimes 1_{<0>} a^i = x,$$

for A is a weak H-module algebra. On the other hand, since $1_{<-1>} \otimes 1_{<0>} = 1_{(1)} \otimes s_A(1_{(2)})$ (cf. proof of Theorem 3.11) we have for all $a \in A$, $c \in C$

$$\begin{aligned}\tilde{\theta} \circ \theta(c \otimes_R a) &= 1_{<-1>} \cdot c \otimes_R 1_{<0>} a = 1_{(1)} \cdot c \otimes_R s_A(1_{(2)}) a \\ &= (1_{(1)} \cdot c) \cdot 1_{(2)} \otimes_R a = S^{-1}(1_{(2)}) 1_{(1)} \cdot c \otimes_R a = c \otimes_R a.\end{aligned}$$

This completes the proof that θ is an isomorphism of A-corings. □

Acknowledgements

T. Brzeziński thanks EPSRC for an Advanced Research Fellowship, S. Caenepeel thanks Department of Mathematics, University of Wales Swansea for hospitality, and G. Militaru thanks the Royal Society for a visiting fellowship.

References

[1] G. Böhm. Doi-Hopf modules over weak Hopf algebras, *Comm. Algebra*, 28:4687–4698, 2000.

[2] G. Böhm and K. Szlachányi. A coassociative C^*-quantum group with nonintegral dimension. *Lett. Math. Phys.* 35:437–456, 1996.

[3] G. Böhm, F. Nill and K. Szlachányi. Weak Hopf algebras I. Integral theory and C^*-structure. *J. Algebra* 221:385–438, 1999.

[4] T. Brzeziński. On modules associated to coalgebra-Galois extensions, *J. Algebra*, 215:290–317, 1999.

[5] T. Brzeziński. The structure of corings. Induction functor, Maschke-type theorem and Frobenius and Galois-type properties, *Preprint* math.RA/0002105. To appear in *Alg. Rep. Theory*.

[6] T. Brzeziński and S. Majid. Coalgebra bundles, *Commun. Math. Phys.*, 191:467–492, 1998.

[7] T. Brzeziński and G. Militaru. Bialgebroids, $\times_A$-bialgebras and duality, *Preprint* math.QA/0012164, 2000.

[8] S. Caenepeel and E. De Groot. Modules over weak entwining structures, *Contemp. Mathematics*, 267:31–54, 2000.
[9] S. Caenepeel, Bogdan Ion, G. Militaru and Shenglin Zhu. Separable functors for the category of Doi-Hopf modules. Applications, *Adv. Math.*, 145:239–290, 1999.
[10] S. Caenepeel, G. Militaru and Shenglin Zhu, Doi-Hopf modules, Yetter-Drinfel'd modules and Frobenius type properties, *Trans. Amer. Math. Soc.*, 349:4311–4342, 1997.
[11] Y. Doi. Unifying Hopf modules, *J. Algebra*, 153:373–385, 1992.
[12] P. Etingof and D. Nikshych. Dynamical quantum groups at roots of 1, *Preprint* math.QA/0003221, 2000.
[13] M. Koppinen. Variations on the smash product with applications to group-graded rings, *J. Pure Appl. Alg.* , 104:61–80, 1994.
[14] J.H. Lu. Hopf algebroids and quantum groupoids, *Int. J. Math.*, 7:47–70, 1996.
[15] F. Nill. Axioms for a weak bialgebra, *Preprint* math.QA/9805104.
[16] P. Schauenburg. Bialgebras over noncommutative rings and a structure theorem for Hopf bimodules, *Applied Categorical Structures* 6:193-222, 1998.
[17] M. Sweedler, The predual theorem to the Jacobson-Bourbaki theorem, *Trans. Amer. Math. Soc.* 213:391–406, 1975.
[18] M. Sweedler, Groups of simple algebras, *I.H.E.S, Publ.*, 44:79–89, 1975
[19] P. Szlachányi, Finite quantum groupoids and inclusion of finite type, *Fields Institute Communications*, to appear; QA/0011036, 2000.
[20] M. Takeuchi, Groups of algebras over $A \otimes \bar{A}$, *J. Math. Soc. Japan*, 29:459–492, 1977.

Department of Mathematics, University of Wales Swansea, Singleton Park, Swansea SA2 8PP, U.K.
E-mail address: `T.Brzezinski@swansea.ac.uk`
URL: `http//www-maths.swan.ac.uk/staff/tb`

Faculty of Applied Sciences, Free University of Brussels, VUB, Pleinlaan 2, B-1050 Brussels, Belgium
E-mail address: `scaenepe@vub.ac.be`

Faculty of Mathematics, University of Bucharest, Str. Academiei 14, RO-70109 Bucharest 1, Romania
E-mail address: `gmilit@al.math.unibuc.ro`